ON THE CONTINUOUS AND SMOOTH FIT PRINCIPLE FOR OPTIMAL STOPPING PROBLEMS IN SPECTRALLY NEGATIVE LÉVY MODELS

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ABSTRACT. This paper is concerned with a class of infinite-time horizon optimal stopping problems for spectrally negative Lévy processes. Focusing on strategies of threshold type, we write explicit expressions for the corresponding expected payoff via the scale function, and further pursue optimal candidate threshold levels. We obtain and show the equivalence of the continuous/smooth fit condition and the first-order condition for maximization over threshold levels. This together with problem-specific information about the payoff function can prove optimality over all stopping times. As examples, we give a short proof of the McKean optimal stopping problem (perpetual American put option) and solve an extension to Egami and Yamazaki [17].

Key words: Optimal stopping; Spectrally negative Lévy processes; Scale functions; Continuous and smooth fit
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1. INTRODUCTION

Optimal stopping problems arise in various areas ranging from the classical sequential testing/change-point detection problems to applications in finance. Although all formulations reduce to the problem of maximizing/minimizing the expected payoff over a set of stopping times, the solution methods are mostly problem-specific; they depend significantly on the underlying process, payoff function and time-horizon. This paper pursues a common tool for the class of *infinite-time horizon* optimal stopping problems for *spectrally negative Lévy processes*, or Lévy processes with only negative jumps.

By extending the classical continuous diffusion model to the Lévy model, one can achieve richer and more realistic models. In mathematical finance, the continuity of paths is empirically rejected and cannot explain, for example, the volatility smile and non-zero credit spreads for short-maturity corporate bonds. These issues can often be alleviated by introducing jumps; see, e.g. [12, 20]. Naturally, however, the optimal stopping problem becomes more challenging and cannot enjoy a number of results obtained under the continuity of paths. In the case of one-dimensional continuous diffusion, a full characterization of the value function is known and some practical methods have been developed (see e.g. [2, 13, 14]). Most of these results rely heavily on the continuity assumption; once jumps are involved, only problem-specific approaches are currently available.

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Despite these differences, there exists a common tool known as the *scale function* for both continuous diffusion and spectrally negative Lévy processes. The scale function for the former enables one to transform a problem of any arbitrary diffusion process to that of a standard Brownian motion. For the latter, the scale function has been playing a central role in expressing fluctuation identities for a general spectrally negative Lévy process (see [8, 21]). By taking advantage of the potential measure that can be expressed using the scale function, one can obtain the overshoot distribution at the first exit time, which is generally a big hurdle that typically makes the problem intractable.

The objective of this paper is to pursue, with the help of the scale function, a common technique for the class of optimal stopping problems for spectrally negative Lévy processes. Focusing on the first time it down-crosses a fixed threshold, we express the corresponding expected payoff in terms of the scale function. This semi-explicit form enables us to differentiate and take limits thanks to the smoothness and asymptotic properties of the scale function as obtained, for example, in [11, 21]. By differentiating the expected payoff with respect to the threshold level, we obtain the *first-order condition* as well as the candidate optimal level that makes it vanish. We also obtain the *continuous/smooth fit condition* when the process is of bounded variation or when it contains a diffusion component. These conditions are in fact equivalent and can be obtained generally under mild conditions.

The spectrally negative Lévy model has been drawing much attention recently as a generalization of the classical Black-Scholes model in mathematical finance and also as a generalization of the Cramér-Lundberg model in insurance. A number of authors have succeeded in extending the classical results to the spectrally negative Lévy model by way of scale functions. We refer the reader to [6, 7] for stochastic games, [5, 22, 25] for the optimal dividend problem, [1, 4] for American and Russian options, and [15, 23, 24] for credit risk. In particular, Egami and Yamazaki [17] modeled and obtained the optimal timing of capital reinforcement. As an application of the results obtained in this paper, we give a short proof of the McKean optimal stopping (perpetual American put option) problem with additional running rewards, as well as an extension and its analytical solution to [17].

The rest of the paper is organized as follows. In Section 2, we review the optimal stopping problem for spectrally negative Lévy processes, and then express the expected value corresponding to the first down-crossing time in terms of the scale function. In Section 3, we obtain the first-order condition as well as the continuous/smooth fit condition and show their equivalence. In Section 4, we solve the McKean optimal stopping problem and an extension to [17]. We conclude the paper in Section 5.

2. The Optimal Stopping Problem for Spectrally Negative Lévy Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a spectrally negative Lévy process $X = \{X_t : t \geq 0\}$ characterized uniquely by the *Laplace exponent*

$$\psi(\beta) := \mathbb{E}^0\left[e^{\beta X_1}\right] = c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(0,\infty)} (e^{-\beta z} - 1 + \beta z 1_{\{0<z<1\}})\,\Pi(\mathrm{d}z), \quad \beta \in \mathbb{R}, \tag{2.1}$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty. \tag{2.2}$$

Here $\mathbb{P}^x$ is the conditional probability where $X_0 = x \in \mathbb{R}$ and $\mathbb{E}^x$ is its expectation. It is well-known that ψ is zero at the origin, convex on $\mathbb{R}_+$ and has a right-continuous inverse:

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \quad q \geq 0.$$

In particular, when

$$\int_{(0,\infty)} (1 \wedge z)\, \Pi(\mathrm{d}z) < \infty, \tag{2.3}$$

we can rewrite

$$\psi(\beta) = \mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(0,\infty)} (e^{-\beta z} - 1)\, \Pi(\mathrm{d}z), \quad \beta \in \mathbb{R}$$

where

$$\mu := c + \int_{(0,1)} z\, \Pi(\mathrm{d}z).$$

The process has paths of bounded variation if and only if $\sigma = 0$ and (2.3) holds. It is also assumed that X is not a negative subordinator (decreasing a.s.). Namely, we require μ to be strictly positive if $\sigma = 0$ and (2.3) holds.

Let $\mathbb{F}$ be the filtration generated by X and $\mathcal{S}$ be a set of $\mathbb{F}$-stopping times. We shall consider a general optimal stopping problem of the form:

$$u(x) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-q\tau} g(X_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-qt} h(X_t) \mathrm{d}t \right], \quad x \in \mathbb{R} \tag{2.4}$$

for some discount factor $q > 0$ and locally-bounded measurable functions $g, h : \mathbb{R} \mapsto \mathbb{R}$ which represent, respectively, the payoff received at a given stopping time τ and the running reward up to τ.

Typically, its optimal stopping time is given by the *first down-crossing time* of the form

$$\tau_A := \inf\{t > 0 : X_t \leq A\}, \quad A \in \mathbb{R}. \tag{2.5}$$

Let us denote the corresponding expected payoff by

$$u_A(x) := \mathbb{E}^x \left[e^{-q\tau_A} g(X_{\tau_A}) 1_{\{\tau_A < \infty\}} + \int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right], \quad x, A \in \mathbb{R},$$

which can be decomposed into

$$u_A(x) = \begin{cases} \Gamma_1(x; A) + \Gamma_2(x; A) + \Gamma_3(x; A), & x > A, \\ g(x), & x \leq A, \end{cases}$$

where, for every $x > A$,

$$\begin{aligned} \Gamma_1(x; A) &:= g(A)\mathbb{E}^x \left[e^{-q\tau_A} \right], \\ \Gamma_2(x; A) &:= \mathbb{E}^x \left[e^{-q\tau_A} (g(X_{\tau_A}) - g(A)) 1_{\{X_{\tau_A} < A,\, \tau_A < \infty\}} \right], \\ \Gamma_3(x; A) &:= \mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} h(X_t) \mathrm{d}t \right]. \end{aligned} \tag{2.6}$$

Shortly below, we express each term via the scale function.

Remark 2.1. *This paper does not consider the* first up-crossing time *defined by* $\tau_B^+ := \inf\{t > 0 : X_t \geq B\}$ *because, for the spectrally negative Lévy case, the process always creeps upward* ($g(X_{\tau_B^+}) = g(B)$ *a.s. on* $\{\tau_B^+ < \infty\}$)*, and the expression of the expected value is much simplified. We focus on a more interesting and challenging case where the optimal stopping time is conjectured to be a first down-crossing time.*

2.1. **Scale functions.** Associated with every spectrally negative Lévy process, there exists a (q-)scale function

$$W^{(q)} : \mathbb{R} \mapsto \mathbb{R}; \quad q \geq 0,$$

that is continuous and strictly increasing on $[0, \infty)$ and is uniquely determined by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q).$$

Fix $a > x > 0$. If τ_a^+ is the first time the process goes above a and τ_0 is the first time it goes below zero as a special case of (2.5), then we have

$$\mathbb{E}^x\left[e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_0,\, \tau_a^+ < \infty\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)} \quad \text{and} \quad \mathbb{E}^x\left[e^{-q\tau_0} 1_{\{\tau_a^+ > \tau_0,\, \tau_0 < \infty\}}\right] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)},$$

where

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}.$$

Here we have

$$W^{(q)}(x) = 0 \quad \text{on} \quad (-\infty, 0) \quad \text{and} \quad Z^{(q)}(x) = 1 \quad \text{on} \quad (-\infty, 0]. \tag{2.7}$$

We also have

$$\mathbb{E}^x\left[e^{-q\tau_0}\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \quad x > 0. \tag{2.8}$$

In particular, $W^{(q)}$ is continuously differentiable on $(0, \infty)$ if Π does not have atoms and $W^{(q)}$ is twice-differentiable on $(0, \infty)$ if $\sigma > 0$; see, e.g., [11]. Throughout this paper, we assume the former.

Assumption 2.1. *We assume that* Π *does not have atoms.*

Fix $q > 0$. The scale function increases exponentially;

$$W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} \quad \text{as } x \uparrow \infty. \tag{2.9}$$

There exists a (scaled) version of the scale function $W_{\Phi(q)} = \{W_{\Phi(q)}(x); x \in \mathbb{R}\}$ that satisfies

$$W_{\Phi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x), \quad x \in \mathbb{R} \tag{2.10}$$

and

$$\int_0^\infty e^{-\beta x} W_{\Phi(q)}(x) \mathrm{d}x = \frac{1}{\psi(\beta + \Phi(q)) - q}, \quad \beta > 0.$$

Moreover $W_{\Phi(q)}(x)$ is increasing, and as is clear from (2.9),

$$W_{\Phi(q)}(x) \uparrow \frac{1}{\psi'(\Phi(q))} \quad \text{as } x \uparrow \infty. \tag{2.11}$$

Regarding its behavior in the neighborhood of zero, it is known that

(2.12)
$$W^{(q)}(0) = \left\{ \begin{array}{ll} 0, & \text{unbounded variation} \\ \frac{1}{\mu}, & \text{bounded variation} \end{array} \right\} \quad \text{and} \quad W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \text{ and } \Pi(0,\infty) = \infty \\ \frac{q+\Pi(0,\infty)}{\mu^2}, & \text{compound Poisson} \end{array} \right\};$$

see Lemmas 4.3-4.4 of [23].

For a comprehensive account of the scale function, see [8, 9, 21, 23]. See [16, 31] for numerical methods for computing the scale function.

2.2. **Expressing the expected payoff using the scale function.** We now express (2.6) in terms of the scale function. For the rest of the paper, because $q > 0$, we must have $\Phi(q) > 0$.

First, the following is immediate by (2.8).

Lemma 2.1. *For every $x > A$, we have*
$$\Gamma_1(x;A) = g(A)\left[Z^{(q)}(x-A) - \frac{q}{\Phi(q)} W^{(q)}(x-A)\right].$$

For Γ_2 and Γ_3, we use the potential measure written in terms of the scale function. By using Theorem 1 of [9] (see also [18, 30]), we have, for every $B \in \mathcal{B}(\mathbb{R})$ and $a > x > A$,
$$\mathbb{E}^x\left[\int_0^{\tau_A \wedge \tau_a^+} e^{-qt} 1_{\{X_t \in B\}} \mathrm{d}t\right] = \int_{B \cap [A,\infty)} \left[\frac{W^{(q)}(x-A) W^{(q)}(a-y)}{W^{(q)}(a-A)} - 1_{\{x \geq y\}} W^{(q)}(x-y)\right] \mathrm{d}y.$$
By taking $a \uparrow \infty$ via the dominated convergence theorem, we can obtain $\Gamma_3(x;A)$ in (2.6). For the problem to be well-defined, we assume throughout the paper the following so that Γ_3 is finite. For a complete proof of Lemma 2.2 below, see [17].

Assumption 2.2. *We assume that $\int_0^\infty e^{-\Phi(q)y}|h(y)|\mathrm{d}y < \infty$.*

Lemma 2.2. *For all $x > A$, we have*
$$\Gamma_3(x;A) = W^{(q)}(x-A)\int_0^\infty e^{-\Phi(q)y} h(y+A)\mathrm{d}y - \int_A^x W^{(q)}(x-y)h(y)\mathrm{d}y.$$

For Γ_2, we first define, for every $A \in \mathbb{R}$,

(2.13)
$$\begin{aligned} \rho_{g,A}^{(q)} &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi(q)z}(g(z+A-u) - g(A))\mathrm{d}z \\ &\equiv \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} e^{-\Phi(q)(y-A)}(g(y-u) - g(A))\mathrm{d}y, \\ \overline{\rho}_{g,A}^{(q)} &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi(q)z}|g(z+A-u) - g(A)|\mathrm{d}z \\ &\equiv \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} e^{-\Phi(q)(y-A)}|g(y-u) - g(A)|\mathrm{d}y. \end{aligned}$$

Lemma 2.3. *Fix $A \in \mathbb{R}$. Suppose*

(1) *g is C^2 in some neighborhood of A and*

(2) *g satisfies*

$$\int_1^\infty \Pi(\mathrm{d}u) \max_{A-u\le\zeta\le A} |g(\zeta)-g(A)| < \infty, \tag{2.14}$$

then $\overline{\rho}_{g,A}^{(q)} < \infty$.

Proof. See Appendix A.1. □

For every $x > A$, we also define

$$\begin{aligned}
\varphi_{g,A}^{(q)}(x) &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))\mathrm{d}z, \\
\overline{\varphi}_{g,A}^{(q)}(x) &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)|g(z+A-u)-g(A)|\mathrm{d}z.
\end{aligned}$$

By (2.10)-(2.11),

$$\begin{aligned}
\overline{\varphi}_{g,A}^{(q)}(x) &= e^{\Phi(q)(x-A)} \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} e^{-\Phi(q)z} W_{\Phi(q)}(x-z-A)|g(z+A-u)-g(A)|\mathrm{d}z \\
&\le e^{\Phi(q)(x-A)} \frac{\overline{\rho}_{g,A}^{(q)}}{\psi'(\Phi(q))},
\end{aligned} \tag{2.15}$$

and hence the finiteness of $\overline{\rho}_{g,A}^{(q)}$ also implies that of $\overline{\varphi}_{g,A}^{(q)}(x)$ for any $x > A$.

Using these notations, Lemma 2.2 together with the compensation formula shows the following.

Lemma 2.4. *If (1)-(2) of Lemma 2.3 hold for a given $A \in \mathbb{R}$, then*

$$\Gamma_2(x;A) = W^{(q)}(x-A)\rho_{g,A}^{(q)} - \varphi_{g,A}^{(q)}(x), \quad x > A. \tag{2.16}$$

Proof. Let $N(\cdot,\cdot)$ be the Poisson random measure associated with $-X$ and $\underline{X}_t := \min_{0\le s\le t} X_s$ for all $t \ge 0$. We also let $x_+ = \max(x,0)$ and $x_- = \max(-x,0)$ for any $x \in \mathbb{R}$. By the compensation formula (see, e.g., [21]),

$$\begin{aligned}
&\mathbb{E}^x \left[e^{-q\tau_A} (g(X_{\tau_A}) - g(A))_+ 1_{\{X_{\tau_A} < A,\, \tau_A < \infty\}} \right] \\
&= \mathbb{E}^x \left[\int_0^\infty \int_0^\infty N(\mathrm{d}t, \mathrm{d}u) e^{-qt} (g(X_{t-}-u) - g(A))_+ 1_{\{X_{t-}-u \le A,\, \underline{X}_{t-} > A\}} \right] \\
&= \mathbb{E}^x \left[\int_0^\infty e^{-qt} \mathrm{d}t \int_0^\infty \Pi(\mathrm{d}u) (g(X_{t-}-u) - g(A))_+ 1_{\{X_{t-}-u \le A,\, \underline{X}_{t-} > A\}} \right] \\
&= \int_0^\infty \Pi(\mathrm{d}u) \mathbb{E}^x \left[\int_0^\infty e^{-qt} (g(X_{t-}-u) - g(A))_+ 1_{\{X_{t-}-u \le A,\, \underline{X}_{t-} > A\}} \mathrm{d}t \right] \\
&= \int_0^\infty \Pi(\mathrm{d}u) \mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} (g(X_{t-}-u) - g(A))_+ 1_{\{X_{t-} \le A+u\}} \mathrm{d}t \right].
\end{aligned}$$

By setting $h(y) \equiv (g(y-u)-g(A))_+ 1_{\{y \leq A+u\}}$ or equivalently $h(y+A) \equiv (g(y+A-u)-g(A))_+ 1_{\{y \leq u\}}$ in Lemma 2.2,

$$
\begin{aligned}
&\mathbb{E}^x \left[\int_0^{\tau_A} e^{-qt} (g(X_{t-}-u)-g(A))_+ 1_{\{X_{t-} \leq A+u\}} \mathrm{d}t \right] \\
&= W^{(q)}(x-A) \int_0^u e^{-\Phi(q)y} (g(y+A-u)-g(A))_+ \mathrm{d}y - \int_A^x W^{(q)}(x-y)(g(y-u)-g(A))_+ 1_{\{y \leq A+u\}} \mathrm{d}y \\
&= W^{(q)}(x-A) \int_0^u e^{-\Phi(q)y} (g(y+A-u)-g(A))_+ \mathrm{d}y - \int_0^{u \wedge (x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))_+ \mathrm{d}z.
\end{aligned}
$$

By substituting this, we have

$$
\begin{aligned}
&\mathbb{E}^x \left[e^{-q\tau_A} (g(X_{\tau_A})-g(A))_+ 1_{\{X_{\tau_A} < A, \, \tau_A < \infty\}} \right] \\
&= \int_0^\infty \Pi(\mathrm{d}u) \Bigg[W^{(q)}(x-A) \int_0^u e^{-\Phi(q)y} (g(y+A-u)-g(A))_+ \mathrm{d}y \\
&\qquad - \int_0^{u \wedge (x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))_+ \mathrm{d}z \Bigg],
\end{aligned}
$$

which is finite by Lemma 2.3 and (2.15). Similarly, we can obtain $\mathbb{E}^x \left[e^{-q\tau_A} (g(X_{\tau_A})-g(A))_- 1_{\{X_{\tau_A} < A, \, \tau_A < \infty\}} \right]$ and (2.16) is immediate by subtraction. □

In view of (2.16) above, we can also write

$$
\begin{aligned}
W^{(q)}(x-A)\rho_{g,A}^{(q)} &= W_{\Phi(q)}(x-A) e^{\Phi(q)x} \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} e^{-\Phi(q)y} (g(y-u)-g(A)) \mathrm{d}y, \\
\varphi_{g,A}^{(q)}(x) &= \int_0^\infty \Pi(\mathrm{d}u) \int_A^{(u+A) \wedge x} W^{(q)}(x-z)(g(z-u)-g(A)) \mathrm{d}z.
\end{aligned} \tag{2.17}
$$

3. First-Order Condition and Continuous and Smooth fit

The most common way of choosing the candidate threshold level is the continuous and smooth fit principle. Define

$$
u_A(A+) := \lim_{x \downarrow A} u_A(x) \quad \text{and} \quad u_A'(A+) := \lim_{x \downarrow A} u_A'(x), \quad A \in \mathbb{R},
$$

if these limits exist. The continuous and smooth fit chooses A such that $u_A(A+) = g(A)$ and $u_A'(A+) = g'(A)$, respectively. Alternatively, one can differentiate u_A with respect to A and obtain the first-order condition.

In this section, we pursue the candidate threshold level A^* in both ways. We first obtain, for a general case, the first-derivative $\partial u_A(x)/\partial A$ and A that makes it vanish, and then the continuous fit condition for the case X is of bounded variation and the smooth fit condition for the case X has a diffusion component ($\sigma > 0$). We further discuss the equivalence of these conditions and how to obtain optimal strategies.

3.1. **First-order condition.** We shall obtain $\partial u_A(x)/\partial A$ for $x > A$ that satisfies (1)-(2) of Lemma 2.3. Let

$$
\Psi(A) := -\frac{q}{\Phi(q)} g(A) + \rho_{g,A}^{(q)} + \int_0^\infty e^{-\Phi(q)y} h(y+A) \mathrm{d}y, \quad A \in \mathbb{R}.
$$

Proposition 3.1 (derivative of u_A with respect to A)**.** *For given $x > A$, suppose (1)-(2) of Lemma 2.3 hold and*

$$\int_1^\infty \Pi(\mathrm{d}u) \sup_{0 \le \xi \le \delta} |g(A+\xi) - g(A+\xi-u)| < \infty, \tag{3.1}$$

for some $\delta > 0$. Then, we have

$$\frac{\partial}{\partial A} u_A(x) = -\Theta^{(q)}(x-A)\Big(\Psi(A) - \frac{\sigma^2}{2} g'(A)\Big),$$

where

$$\Theta^{(q)}(y) := e^{\Phi(q)y} W'_{\Phi(q)}(y), \quad y > 0.$$

Because $W_{\Phi(q)}$ is increasing, $\Theta^{(q)}$ is positive (see also [23] for an interpretation of $\Theta^{(q)}$ as the resolvent measure of the ascending ladder height process of X) and hence

$$\Psi(A) - \frac{\sigma^2}{2} g'(A) \le (\ge) 0 \Longrightarrow \frac{\partial}{\partial A} u_A(x) \ge (\le) 0 \quad \forall x > A. \tag{3.2}$$

If there exists A^* such that

$$\Psi(A^*) - \frac{\sigma^2}{2} g'(A^*) = 0, \tag{3.3}$$

then the stopping time τ_{A^*} naturally becomes a reasonable candidate for the optimal stopping time.

In order to show Proposition 3.1 above, we obtain the derivatives of Γ_i for $1 \le i \le 3$ with respect to A for any $x > A$. By applying straightforward differentiation in Lemma 2.1 and because $W^{(q)\prime}(x) = \Phi(q) W_{\Phi(q)}(x) + \Theta^{(q)}(x)$,

$$\frac{\partial}{\partial A} \Gamma_1(x; A) = g'(A)\Big[Z^{(q)}(x-A) - \frac{q}{\Phi(q)} W^{(q)}(x-A)\Big] + g(A) \frac{q}{\Phi(q)} \Theta^{(q)}(x-A). \tag{3.4}$$

For Γ_2, we first take the derivatives of (2.17) with respect to A.

Lemma 3.1. *Fix $x > A$. Under the assumptions in Proposition 3.1,*

$$\begin{aligned} \frac{\partial}{\partial A} \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} e^{-\Phi(q)y} [g(y-u) - g(A)] \mathrm{d}y \\ = e^{-\Phi(q)A} \int_0^\infty \Pi(\mathrm{d}u) \Big[g(A) - g(A-u) - \frac{1 - e^{-\Phi(q)u}}{\Phi(q)} g'(A)\Big], \end{aligned} \tag{3.5}$$

and

$$\frac{\partial}{\partial A} \varphi_{g,A}^{(q)}(x) = \int_0^\infty \Pi(\mathrm{d}u) \Big[W^{(q)}(x-A)(g(A) - g(A-u)) - g'(A) \int_A^{(u+A)\wedge x} W^{(q)}(x-z) \mathrm{d}z\Big]. \tag{3.6}$$

Proof. See Appendix A.2. $\square$

By applying Lemma 3.1 in (2.16)-(2.17), the derivative of Γ_2 with respect to A is immediately obtained.

Lemma 3.2. *Fix $x > A$. Under the assumptions in Proposition 3.1,*

$$\begin{aligned} \frac{\partial}{\partial A} \Gamma_2(x; A) = -W'_{\Phi(q)}(x-A) e^{\Phi(q)x} \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} e^{-\Phi(q)y} (g(y-u) - g(A)) \mathrm{d}y \\ + g'(A) \int_0^\infty \Pi(\mathrm{d}u) \Big(\int_A^{(u+A)\wedge x} W^{(q)}(x-z) \mathrm{d}z - \frac{1 - e^{-\Phi(q)u}}{\Phi(q)} W^{(q)}(x-A) \Big). \end{aligned}$$

For Γ_3, as in the proof of Lemma 4.4 of [17], we have the following. Although the continuity of h is assumed throughout in [17], it is not required in the following lemma; this is clear from the proof of Lemma 4.4 of [17].

Lemma 3.3. *For every $x > A$,*

$$\frac{\partial}{\partial A}\Gamma_3(x;A) = -\Theta^{(q)}(x-A)\int_0^\infty e^{-\Phi(q)y}h(y+A).$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. By combining (3.4) and Lemmas 3.2-3.3, we obtain

$$\frac{\partial}{\partial A}u_A(x) = -\Theta^{(q)}(x-A)\Psi(A) + g'(A)Q(x;A)$$

where

$$\begin{aligned} Q(x;A) := {} & Z^{(q)}(x-A) - \frac{q}{\Phi(q)}W^{(q)}(x-A) \\ & - \int_0^\infty \Pi(\mathrm{d}u)\Big(W^{(q)}(x-A)\frac{1-e^{-\Phi(q)u}}{\Phi(q)} - \int_A^{(u+A)\wedge x} W^{(q)}(x-z)\mathrm{d}z\Big), \quad x > A. \end{aligned}$$

By Lemmas 2.1 and modifying Lemma 2.4, we can also write

$$Q(x;A) = \mathbb{E}^x\left[e^{-q\tau_A}\right] - \mathbb{E}^x\left[e^{-q\tau_A}1_{\{X_{\tau_A}<A,\,\tau_A<\infty\}}\right] = \mathbb{E}^x\left[e^{-q\tau_A}1_{\{X_{\tau_A}=A,\,\tau_A<\infty\}}\right], \quad x > A.$$

A spectrally negative Lévy process creeps downward if and only if there is a Gaussian component, i.e., $\mathbb{P}^x\left\{X_{\tau_A} = A\right\} > 0$ for any $x > A$ if and only if $\sigma > 0$; see [21] Exercise 7.6. Hence

$$\sigma > 0 \Longleftrightarrow Q(x;A) > 0, \ \forall x > A.$$

This proves the desired result for the case $\sigma = 0$. For the case $\sigma > 0$, as in [10, 28], we can also write

$$Q(x;A) = \frac{\sigma^2}{2}\left(W^{(q)\prime}(x-A) - \Phi(q)W^{(q)}(x-A)\right) = \frac{\sigma^2}{2}\Theta^{(q)}(x-A),$$

and hence it also holds when $\sigma > 0$ as well. □

3.2. **Continuous and smooth fit.** We now pursue A^* such that $u_{A^*}(A^*+) = g(A^*)$ and $u'_{A^*}(A^*+) = g'(A^*)$ for the cases

(1) X is of bounded variation, and
(2) $\sigma > 0$,

respectively. We exclude the case X is of unbounded variation with $\sigma = 0$ (in this case, $W^{(q)\prime}(0+) = \infty$ by (2.12) and hence the interchange of limits over integrals we conduct below may not be valid). However, this can be alleviated and the results hold generally for all spectrally negative Lévy processes when g is a constant in a neighborhood of A^*. Examples include [17] where $g(x) = 0$ on $(0,\infty)$ and [29] where $g(x) = 1$ on $(-\infty, 0]$ and $g(x) = 2$ on $(0,\infty)$; see Section 4.

For continuous fit, we need to obtain

$$\Gamma_1(A+;A) := \lim_{x\downarrow A}\Gamma_1(x;A), \quad \Gamma_2(A+;A) := \lim_{x\downarrow A}\Gamma_2(x;A), \quad \text{and} \quad \Gamma_3(A+;A) := \lim_{x\downarrow A}\Gamma_3(x;A)$$

if these limits exist. Define also $\varphi_{g,A}^{(q)}(A+) := \lim_{x \downarrow A} \varphi_{g,A}^{(q)}(x)$, if it exists. It is easy to see that

$$\Gamma_1(A+;A) = g(A)\left(1 - \frac{q}{\Phi(q)} W^{(q)}(0)\right) \quad \text{and} \quad \Gamma_3(A+;A) = W^{(q)}(0) \int_0^\infty e^{-\Phi(q)y} h(y+A) \mathrm{d}y. \tag{3.7}$$

The result for Γ_2 is immediate by the dominated convergence theorem thanks to Lemma 2.3 and (2.15)-(2.16).

Lemma 3.4. *Given (1)-(2) of Lemma 2.3 for a given $A \in \mathbb{R}$, we have*

(1) $\varphi_{g,A}^{(q)}(A+) = 0$,
(2) $\Gamma_2(A+;A) = W^{(q)}(0)\rho_{g,A}^{(q)}$.

Now Lemma 3.4 and (3.7) show

$$u_A(A+) = g(A) + W^{(q)}(0)\Psi(A). \tag{3.8}$$

This together with (2.12) shows the following.

Proposition 3.2 (Continuous Fit)**.** *Fix $A \in \mathbb{R}$ and suppose (1)-(2) of Lemma 2.3 hold.*

(1) *If X is of bounded variation, the continuous fit condition $u_A(A+) = g(A)$ holds if and only if*
$$\Psi(A) = 0.$$
(2) *If X is of unbounded variation (including the case $\sigma = 0$), it is automatically satisfied.*

For the case X is of unbounded variation with $\sigma > 0$, we shall pursue smooth fit condition at $A \in \mathbb{R}$. The following lemma says in this case that the derivative can go into the integral sign and we can further interchange the limit.

Lemma 3.5. *Fix $A \in \mathbb{R}$. If $\sigma > 0$ and suppose (1)-(2) of Lemma 2.3 hold, then*

$$\varphi_{g,A}^{(q)\prime}(x) = \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u \wedge (x-A)} W^{(q)\prime}(x-z-A)[g(z+A-u) - g(A)]\mathrm{d}z, \quad x > A, \tag{3.9}$$

and

$$\varphi_{g,A}^{(q)\prime}(A+) = 0. \tag{3.10}$$

Proof. See Appendix A.3. □

Remark 3.1. *In the case of unbounded variation with $\sigma = 0$, it is expected that* (3.9) *holds but* (3.10) *does not. This is because $W^{(q)\prime}(0+) = \infty$ and the limit cannot go into the integral.*

We are now ready to obtain $\Gamma_i'(A+;A)$ for $1 \le i \le 3$.

Lemma 3.6. *Fix $A \in \mathbb{R}$. Suppose $\sigma > 0$ and (1)-(2) of Lemma 2.3 hold. Then,*

(1) $\Gamma_1'(A+,A) = -W^{(q)\prime}(0+)g(A)q/\Phi(q)$,
(2) $\Gamma_2'(A+;A) = W^{(q)\prime}(0+)\rho_{g,A}^{(q)}$,
(3) $\Gamma_3'(A+;A) = W^{(q)\prime}(0+)\int_0^\infty e^{-\Phi(q)y} h(y+A)\mathrm{d}y$.

Proof. (1) It is immediate by Lemma 2.1. (2) By (2.16),

$$\Gamma_2'(x;A) = W^{(q)\prime}(x-A)\rho_{g,A}^{(q)} - \varphi_{g,A}^{(q)\prime}(x), \quad x > A.$$

By taking $x \downarrow A$ via (3.10), we have the claim. (3) Using (2.7) in particular $W^{(q)}(0) = 0$, we have

$$\begin{aligned}\Gamma_3'(A+;A) &= \lim_{x \downarrow A}\left[W^{(q)\prime}(x-A)\int_0^\infty e^{-\Phi(q)y}h(y+A)\mathrm{d}y - \int_A^x W^{(q)\prime}(x-y)h(y)\mathrm{d}y\right] \\ &= W^{(q)\prime}(0+)\int_0^\infty e^{-\Phi(q)y}h(y+A)\mathrm{d}y.\end{aligned}$$

□

By the lemma above, we obtain

$$u_A'(A+) = W^{(q)\prime}(0+)\Psi(A)$$

or equivalently, by virtue of (2.12), the smooth fit condition at A^* is equivalent to (3.3).

Proposition 3.3 (Smooth Fit)**.** *Fix $A \in \mathbb{R}$. Suppose $\sigma > 0$ and (1)-(2) of Lemma 2.3 hold. Then, the smooth fit condition $u_A'(A+) = g'(A)$ holds if and only if*

$$\Psi(A) = \frac{\sigma^2}{2}g'(A).$$

We summarize the results obtained in Propositions 3.2-3.3 in Table 1. It is clear from Proposition 3.1 and Table 1 that the first-order condition and the continuous/smooth fit condition are indeed equivalent.

	Continuous-fit	Smooth-fit
bounded var.	$\Psi(A) = 0$	N/A
$\sigma > 0$	Automatically satisfied	$\Psi(A) = \sigma^2 g'(A)/2$

TABLE 1. Summary of Continuous- and Smooth-fit Conditions.

3.3. **Obtaining optimal solution.** After choosing the candidate threshold level A^*, the verification of optimality of τ_{A^*} requires

(i) $u_{A^*}(x) \geq g(x)$ for all $x \in \mathbb{R}$,
(ii) $(\mathcal{L} - q)u_{A^*}(x) + h(x) = 0$ for all $x \in (A^*, \infty)$,
(iii) $(\mathcal{L} - q)u_{A^*}(x) + h(x) < 0$ for all $x \in (-\infty, A^*)$;

see e.g., [27]. Here $\mathcal{L}$ is the infinitesimal generator associated with the process X applied to sufficiently smooth function f

$$\mathcal{L}f(x) = cf'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_0^\infty \left[f(x-z) - f(x) + f'(x)z1_{\{0<z<1\}}\right]\Pi(\mathrm{d}z).$$

As we shall show shortly below, the conditions (i)-(ii) can be obtained upon some conditions. The proof of condition (iii) unfortunately relies on the structure of the problem; in order to complement this, we give examples where the optimality over all stopping times holds in the next section.

Lemma 3.7. *Suppose A^* satisfies (3.3), g is C^2 on (A^*, ∞) and*

$$\Psi(A) - \frac{\sigma^2}{2} g'(A) > 0, \qquad A > A^*. \tag{3.11}$$

Then (i) is satisfied.

Proof. Because $g(x) = u_{A^*}(x)$ on $(-\infty, A^*]$, we only need to show (i) on (A^*, ∞). For any $x > A^*$, we obtain by (3.2) and (3.8) that

$$u_{A^*}(x) \geq \lim_{A \uparrow x} u_A(x) = g(x) + W^{(q)}(0)\Psi(x).$$

For the unbounded variation case, because $W^{(q)}(0) = 0$, the result is immediate. For the bounded variation case (which necessarily means $\sigma = 0$), (3.11) implies $\Psi(x) > 0$ and hence the result is also immediate.

□

Regarding the condition (ii), the stochastic processes

$$\left\{e^{-q(t \wedge \tau_B^+ \wedge \tau_0)} W^{(q)}(X_{t \wedge \tau_B^+ \wedge \tau_0}); t \geq 0\right\} \quad \text{and} \quad \left\{e^{-q(t \wedge \tau_B^+ \wedge \tau_0)} Z^{(q)}(X_{t \wedge \tau_B^+ \wedge \tau_0}); t \geq 0\right\}$$

for any $B < \infty$ are martingales (see, e.g., [10]), and therefore

$$(\mathcal{L} - q)W^{(q)}(x) = (\mathcal{L} - q)Z^{(q)}(x) = 0, \quad x > 0. \tag{3.12}$$

Furthermore, integration by parts can be applied to obtain the following (see Section A.5 of [17] for a complete proof).

Lemma 3.8 (Egami and Yamazaki [17])**.** *We have*

$$(\mathcal{L} - q)\left[\int_A^x W^{(q)}(x-y)h(y)\mathrm{d}y\right] = h(x), \quad x > A.$$

By Lemma 3.8, we obtain the following.

Proposition 3.4. *For every $x > A$, we have $(\mathcal{L} - q)u_A(x) + h(x) = 0$.*

Proof. Define $f(x) := \mathbb{E}^x\left[e^{-q\tau_A} g(X_{\tau_A})\right]$. Then for all $x > A$, we have by the strong Markov property,

$$\mathbb{E}^x\left[e^{-q\tau_A} g(X_{\tau_A}) | \mathcal{F}_{t \wedge \tau_A}\right] = e^{-q(t \wedge \tau_A)} f(X_{t \wedge \tau_A}).$$

Taking expectation on both sides we obtain $f(x) = \mathbb{E}^x\left[e^{-q\tau_A} g(X_{\tau_A})\right] = \mathbb{E}^x\left[e^{-q(t \wedge \tau_A)} f(X_{t \wedge \tau_A})\right]$. Hence $\left\{e^{-q(t \wedge \tau_A)} f(X_{t \wedge \tau_A}); t \geq 0\right\}$ is a martingale and therefore $(\mathcal{L} - q)f(x) = (\mathcal{L} - q)(\Gamma_1(x; A) + \Gamma_2(x; A)) = 0$ on (A, ∞); see also the appendix of [10] for a more rigorous proof. On the other hand, Lemma 3.8 and (3.12) give

$$(\mathcal{L} - q)\Gamma_3(x; A) = -(\mathcal{L} - q)\left[\int_A^x W^{(q)}(x-y)h(y)\mathrm{d}y\right] = -h(x).$$

Summing up these, we have the claim. □

4. Examples

In this section, we give examples to illustrate how we can apply the results obtained in the previous sections. We first consider, as a warm-up, a generalized version of the McKean optimal stopping problem with additional running rewards. We then extend Egami and Yamazaki [17] and obtain analytical solutions.

4.1. **The McKean optimal stopping.** The classical McKean optimal stopping problem, also known as the pricing of a perpetual American put option, reduces to (2.4) with $g(x) = K - e^x$ and $h \equiv 0$. Here, e^X models the stock price and $K > 0$ is the strike price; the option holder chooses a time to exercise so as to maximize the expected payoff. In particular, for the spectrally negative case, it has been shown by [3] that the optimal threshold level is given by

$$A^* = \log\Big(K \frac{q}{\Phi(q)} \frac{\Phi(q)-1}{q-\psi(1)}\Big). \tag{4.1}$$

We consider a more general case where h is any non-decreasing function and give a simple proof by directly using the results obtained in the previous sections. Here we assume that $\psi(1) \neq q$ (or $\Phi(q) \neq 1$); the case of $\psi(1) = q$ can be obtained by taking limits on the results described below. Because

$$\begin{aligned}
-g(A)\frac{q}{\Phi(q)} + \rho^{(q)}_{g,A} &= -\frac{q}{\Phi(q)}(K - e^A) + \int_0^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi(q)z}(e^A - e^{z+A-u})\mathrm{d}z \\
&= -\frac{q}{\Phi(q)}K + e^A\Big[\frac{q}{\Phi(q)} + \int_0^\infty \Pi(\mathrm{d}u)\Big(\frac{1-e^{-\Phi(q)u}}{\Phi(q)} - e^{-u}\frac{1-e^{-(\Phi(q)-1)u}}{\Phi(q)-1}\Big)\Big],
\end{aligned}$$

we obtain

$$\Psi(A) - \frac{\sigma^2}{2}g'(A) = -\frac{q}{\Phi(q)}K + \frac{e^A}{\Phi(q)}M_q + \int_0^\infty e^{-\Phi(q)y}h(y+A)\mathrm{d}y \tag{4.2}$$

where

$$M_q := q + \frac{\sigma^2}{2}\Phi(q) + \int_0^\infty \Pi(\mathrm{d}u)\Big[(1 - e^{-\Phi(q)u}) - e^{-u}(1 - e^{-(\Phi(q)-1)u})\frac{\Phi(q)}{\Phi(q)-1}\Big].$$

Here, by the change of measure, M_q can be simplified.

Lemma 4.1. *We have* $M_q = \frac{\Phi(q)}{\Phi(q)-1}(q - \psi(1))$.

Proof. By the definition of ψ and Φ, we rewrite M_q as

$$\begin{aligned}
&q + \frac{\sigma^2}{2}\Phi(q) + \int_0^\infty \Pi(\mathrm{d}u)\left[(1 - e^{-\Phi(q)u} - \Phi(q)u1_{\{u\in(0,1)\}}) - e^{-u}(1 - e^{-(\Phi(q)-1)u})\frac{\Phi(q)}{\Phi(q)-1} + \Phi(q)u1_{\{u\in(0,1)\}}\right] \\
&= \Big(c - \int_0^1 u(e^{-u}-1)\Pi(\mathrm{d}u)\Big)\Phi(q) + \frac{\sigma^2}{2}\Phi(q)(\Phi(q)+1) \\
&\quad - \frac{\Phi(q)}{\Phi(q)-1}\int_0^\infty \Pi(\mathrm{d}u)e^{-u}\left(1 - e^{-(\Phi(q)-1)u} + (\Phi(q)-1)u1_{\{u\in(0,1)\}}\right).
\end{aligned}$$

Define, as the Laplace exponent of X under $\mathbb{P}_1$ with the change of measure $\frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp(X_t - \psi(1)t)$, $t \geq 0$,

$$\psi_1(\beta) := \Big(\sigma^2 + c - \int_0^1 u(e^{-u}-1)\Pi(\mathrm{d}u)\Big)\beta + \frac{1}{2}\sigma^2\beta^2 + \int_0^\infty (e^{-\beta u} - 1 + \beta u1_{\{u\in(0,1)\}})e^{-u}\,\Pi(\mathrm{d}u).$$

Then, $\psi_1(\Phi(q)-1) = \psi(\Phi(q)) - \psi(1) = q - \psi(1)$; see page 215 of [21]. Hence simple algebra shows

$$\frac{\Phi(q)}{\Phi(q)-1}(q - \psi(1)) = \frac{\Phi(q)}{\Phi(q)-1}\psi_1(\Phi(q)-1) = M_q,$$

as desired. □

It is clear that $M_q > 0$ and hence (4.2) is monotonically increasing in A. Recall also Assumption 2.2. Therefore on condition that

$$\lim_{A \downarrow -\infty} \left[\Psi(A) - \frac{\sigma^2}{2} g'(A) \right] = -\frac{q}{\Phi(q)} K + \lim_{A \downarrow -\infty} \int_0^\infty e^{-\Phi(q) y} h(y+A) \mathrm{d}y < 0, \tag{4.3}$$

there exists a unique A^* such that (4.2) vanishes and by (3.2)

$$\frac{\partial}{\partial A} u_A(x) \geq 0 \quad \forall x > A \Longleftrightarrow A \leq A^*. \tag{4.4}$$

This shows that τ_{A^*} is optimal among the set of all stopping times of threshold type. Notice in a special case when $h \equiv 0$, the optimal threshold A^* reduces to (4.1). Because the optimal stopping time is known to be of threshold type by [26], τ_{A^*} is indeed the optimal stopping time.

We now show that it is indeed optimal over all stopping times even when h is not zero. This reduces to showing (iii) (in Section 3.3) because (i) holds thanks to (4.4) and Lemma 3.7 and (ii) thanks to Proposition 3.4. For this special case of g, we can simplify as in Exercise 8.7 (ii) and Corollary 9.3 of [21] for any $x, A \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}^x \left[e^{-q\tau_A} (K - e^{X_{\tau_A}}) \right] &= K \Big(Z^{(q)}(x-A) - \frac{q}{\Phi(q)} W^{(q)}(x-A) \Big) \\ &\quad - e^x \Big(Z_1^{(q-\psi(1))}(x-A) - \frac{q-\psi(1)}{\Phi(q)-1} W_1^{(q-\psi(1))}(x-A) \Big) \end{aligned} \tag{4.5}$$

where W_1 and Z_1 are versions of W and Z associated with the measure $\mathbb{P}^1$ under the same change of measure as in the proof of Lemma 4.1. Here, notice as in Lemmas 8.3 and 8.5 of [21], for each $x > 0$, the functions $q \mapsto W^{(q)}(x)$ and $q \mapsto Z^{(q)}(x)$ can be analytically extended to $q \in \mathbb{C}$. In particular, by Lemma 8.4 of [21],

$$e^x W_1^{(q-\psi(1))}(x) = W^{(q)}(x), \quad x \geq 0. \tag{4.6}$$

Proposition 4.1. *Suppose h is non-decreasing and (4.3) holds. Then there exists a unique A^* such that (4.2) vanishes. Moreover, τ_{A^*} is an optimal stopping time and the optimal value function is given by*

$$u(x) = K Z^{(q)}(x-A^*) - e^x Z_1^{(q-\psi(1))}(x-A^*) - \int_{A^*}^x W^{(q)}(x-y) h(y) \mathrm{d}y.$$

Proof. By Lemma 4.1 and the discussion above this proposition, there exists a unique A^* such that

$$0 = -qK + e^{A^*} \frac{\Phi(q)}{\Phi(q)-1} (q - \psi(1)) + \Phi(q) \int_0^\infty e^{-\Phi(q) y} h(y + A^*) \mathrm{d}y. \tag{4.7}$$

We first show (iii). By (2.1), $\mathcal{L} g(x) = -e^x \left[c + \frac{1}{2}\sigma^2 + \int_0^\infty \left[e^{-z} - 1 + z 1_{\{0<z<1\}} \right] \Pi(\mathrm{d}z) \right] = -e^x \psi(1)$, and hence

$$(\mathcal{L} - q) g(x) + h(x) = -qK + e^x (q - \psi(1)) + h(x). \tag{4.8}$$

Because h is non-decreasing and $x < A^*$

$$\Phi(q) \int_0^\infty e^{-\Phi(q) y} h(y + A^*) \mathrm{d}y \geq \Phi(q) \int_0^\infty e^{-\Phi(q) y} h(x) \mathrm{d}y = h(x). \tag{4.9}$$

It is also easy to see that

$$e^{A^*} \frac{\Phi(q)}{\Phi(q)-1} (q - \psi(1)) \geq e^x (q - \psi(1)). \tag{4.10}$$

Indeed, for the case $q-\psi(1)>0$, we must have $\Phi(q)-1>0$ and hence (4.10) holds by $A^*>x$; for the case $q-\psi(1)<0$, the left hand side is positive while the right hand side is negative in (4.10). By (4.8)-(4.10), (iii) holds.

Now by the continuous and smooth fit results obtained in the previous section, u is C^1 on $\mathbb{R}\backslash\{A^*\}$ and continuous at A^* (with locally bounded derivative around A^*) when $\sigma=0$, while it is C^2 on $\mathbb{R}\backslash\{A^*\}$ and differentiable at A^* (with locally bounded second derivative around A^*) when $\sigma>0$. These together with (i) and (ii) show the optimality using a standard technique of optimal stopping; see e.g. [27] Theorem 2.2.

Finally, by (4.5)-(4.7),

$$\begin{aligned} u(x) &= KZ^{(q)}(x-A^*) - e^x\left(Z_1^{(q-\psi(1))}(x-A^*) - W_1^{(q-\psi(1))}(x-A^*)\frac{q-\psi(1)}{\Phi(q)-1}\right) \\ &\quad - W^{(q)}(x-A^*)e^{A^*}\frac{q-\psi(1)}{\Phi(q)-1} - \int_{A^*}^x W^{(q)}(x-y)h(y)\mathrm{d}y \\ &= KZ^{(q)}(x-A^*) - e^x Z_1^{(q-\psi(1))}(x-A^*) - \int_{A^*}^x W^{(q)}(x-y)h(y)\mathrm{d}y, \end{aligned}$$

as desired. □

4.2. **Generalization of Egami and Yamazaki [17].** We now solve an extension to [17], where we obtained an alarm system that determines when a bank needs to start enhancing its own capital ratio so as not to violate the capital adequacy requirements. Here X models the bank's net worth or equity capital allocated to its loan/credit business. The problem is to strike the balance between minimizing the chance of violating the net capital requirement and the costs of premature undertaking (or the regret) measured, respectively, by

$$R_x^{(q)}(\tau) := \mathbb{E}^x\left[e^{-q\theta}1_{\{\tau\geq\theta\}}\right] \quad \text{and} \quad H_x^{(q,h)}(\tau) := \mathbb{E}^x\left[1_{\{\tau<\infty\}}\int_\tau^\theta e^{-qt}h(X_t)\mathrm{d}t\right]$$

where h is positive, continuous and increasing and

$$\theta := \inf\{t\geq 0 : X_t\leq 0\}$$

denotes the capital requirement violation time. We want to obtain over the set of stopping times,

$$\mathcal{S} := \{\tau \text{ stopping time} : \tau\leq\theta\ a.s.\}\,, \tag{4.11}$$

an optimal stopping time that minimizes the linear combination of the two costs described above:

$$U_x^{(q,h)}(\tau,\gamma) := R_x^{(q)}(\tau) + \gamma H_x^{(q,h)}(\tau),$$

for some $\gamma>0$. By taking advantage of the property of $\mathcal{S}$, the problem can be reduced to obtaining

$$\inf_{\tau\in\mathcal{S}}\mathbb{E}^x\left[e^{-q\tau}1_{\{X_\tau\leq 0,\tau<\infty\}} + \int_\tau^\theta e^{-qt}h(X_t)\mathrm{d}t\right] = -u(x) + \mathbb{E}^x\left[\int_0^\theta e^{-qt}h(X_t)\mathrm{d}t\right]$$

with

$$u(x) := \sup_{\tau\in\mathcal{S}}\mathbb{E}^x\left[-e^{-q\tau}1_{\{X_\tau\leq 0,\tau<\infty\}} + \int_0^\tau e^{-qt}h(X_t)\mathrm{d}t\right].$$

In other words, the problem reduces to (2.4) with

$$g(x) = \begin{cases} 0, & x > 0, \\ -1, & x \leq 0, \end{cases}$$

and a special set of stopping times defined in (4.11). Egami and Yamazaki [17] solved for double exponential jump diffusion [19] and for a general spectrally negative Lévy process.

We shall consider its extension for a more general g (or more general $R_x^{(q)}(\tau) := -\mathbb{E}^x\left[e^{-q\theta} g(X_\theta) 1_{\{\tau \geq \theta\}}\right]$) by assuming the following.

Specification 4.1. (1) g is negative and increasing on $(-\infty, 0]$ (and zero on $(0,\infty)$);
(2) h is positive, continuous and increasing.

The first assumption on g means that the penalty $|g(X_\theta)|$ increases as the overshoot $|X_\theta|$ increases. The second assumption on h is the same as in [17]; if a bank has a higher capital value, then it naturally has better access to high quality assets.

In this problem, it can be conjectured that there exists a threshold level A^* such that τ_{A^*} is optimal. Here we can rewrite (2.13) for all $A > 0$

$$\rho_{g,A}^{(q)} = \int_A^\infty \Pi(\mathrm{d}u) \int_0^{u-A} e^{-\Phi(q)y} g(y + A - u) \mathrm{d}y,$$
$$\overline{\rho}_{g,A}^{(q)} = \int_A^\infty \Pi(\mathrm{d}u) \int_0^{u-A} e^{-\Phi(q)y} |g(y + A - u)| \mathrm{d}y.$$

This avoids the integration of Π in the neighborhood of zero and hence Lemma 3.5 also holds for the case of unbounded variation with $\sigma = 0$. Now, as a special case of Propositions 3.2-3.3 (noticing $g(A) = g'(A) = 0$ for all $A > 0$), we obtain the following.

Lemma 4.2 (Continuous and Smooth Fit)**.** *Suppose (1)-(2) of Lemma 2.3 for a given $A > 0$.*

continuous fit: *If X is of bounded variation, the continuous fit condition $u(A) = 0$ holds if and only if*

$$\Psi(A) = 0. \tag{4.12}$$

If X is of unbounded variation, it is automatically satisfied.

smooth fit: *If X is of unbounded variation, the smooth fit condition $u'(A) = 0$ holds if and only if (4.12) holds.*

	Continuous-fit	Smooth-fit
(i) bounded var.	$\Psi(A) = 0$	N/A
(ii) unbounded var.	Automatically satisfied	$\Psi(A) = 0$

TABLE 2. Summary of Continuous- and Smooth-fit Conditions.

Under Specification 4.1, there exists at most one $A^* > 0$ that satisfies (4.12) because

$$\Psi'(A) = \int_0^\infty e^{-\Phi(q)y} h'(y+A) \mathrm{d}y + \int_A^\infty \Pi(\mathrm{d}u) \int_0^{u-A} e^{-\Phi(q)y} g'(y+A-u) \mathrm{d}y - \int_A^\infty \Pi(\mathrm{d}u) e^{-\Phi(q)(u-A)} g(0-) > 0. \tag{4.13}$$

Verification of optimality: We let A^* be the unique root of $\Psi(A) = 0$ if it exists and set it zero otherwise. As in the case of the McKean optimal stopping problem, we only need to show (iii) in Section 3.3 because (i) holds by (4.13) and Lemma 3.7 and (ii) by Proposition 3.4.

Lemma 4.3. *If $A^* > 0$, we have $(\mathcal{L} - q)g(x) + h(x) \leq 0$ for every $x \in (0, A^*)$.*

Proof. Because $g(x) = g'(x) = 0$ for every $x > 0$,

$$(\mathcal{L} - q)g(x) + h(x) = \int_x^\infty \Pi(\mathrm{d}u) g(x-u) + h(x), \quad x \in (0, A^*). \tag{4.14}$$

We shall show that this is negative. Because $A^* > 0$, we must have

$$\int_0^\infty e^{-\Phi(q)y} h(y+A^*) \mathrm{d}y + \int_{A^*}^\infty \Pi(\mathrm{d}u) \int_0^{u-A^*} e^{-\Phi(q)y} g(y+A^*-u) \mathrm{d}y = 0,$$

and hence

$$\begin{aligned}
0 &\geq h(x) \int_0^\infty e^{-\Phi(q)y} \mathrm{d}y + \int_{A^*}^\infty \Pi(\mathrm{d}u) g(x-u) \int_0^{u-A^*} e^{-\Phi(q)y} \mathrm{d}y \\
&\geq h(x) \int_0^\infty e^{-\Phi(q)y} \mathrm{d}y + \int_{A^*}^\infty \Pi(\mathrm{d}u) g(x-u) \int_0^\infty e^{-\Phi(q)y} \mathrm{d}y \\
&\geq h(x) \int_0^\infty e^{-\Phi(q)y} \mathrm{d}y + \int_x^\infty \Pi(\mathrm{d}u) g(x-u) \int_0^\infty e^{-\Phi(q)y} \mathrm{d}y \\
&= \left(h(x) + \int_x^\infty \Pi(\mathrm{d}u) g(x-u) \right) \int_0^\infty e^{-\Phi(q)y} \mathrm{d}y,
\end{aligned}$$

where the first inequality holds because g and h are increasing and $x < A^*$, the second holds because g is nonpositive, and the third holds because $x < A^*$ and g is nonpositive. This together with (4.14) shows the result. □

For the rest of the proof, we refer the reader to the proof of Proposition 4.1 in [17].

Proposition 4.2. *If $A^* > 0$, then τ_{A^*} is the optimal stopping time and the value function is given by $u_{A^*}(x)$ for every $x > 0$. If $A^* = 0$, then the value function is given by $\lim_{A \downarrow 0} u_A(x)$ for every $x > 0$.*

5. Concluding Remarks

We have discussed the optimal stopping problem for spectrally negative Lévy processes. By expressing the expected payoff via the scale function, we achieved the first-order condition as well as the continuous/smooth fit condition and showed their equivalence. The results obtained here can be applied to a wide range of optimal stopping problems for spectrally negative Lévy processes. As examples, we gave a short proof for the perpetual American option pricing problem and solved an extension to Egami and Yamazaki [17].

For future research, it would be interesting to pursue similar results for optimal stopping games. Typically, the equilibrium strategies are given by stopping times of threshold type as in [6, 7, 15] . Similarly to the results obtained in this paper, the expected payoff admits expressions in terms of the scale function and hence the first-order condition and the continuous/smooth fit can be obtained analytically. Another direction is the extension to a general Lévy process with both positive and negative jumps. This can be obtained in terms of the Wiener-Hopf factor alternatively to the scale function. Finally, the results can be extended to a number of variants of optimal stopping such as optimal switching, impulse control and multiple stopping.

Appendix A. Proofs

A.1. **Proof of Lemma 2.3.** By the assumption (1) and Taylor expansion, we can take $0 < \epsilon < 1$ such that, for any $0 < z < u < \epsilon$ and $\varrho_{A,\epsilon} := \max_{0 \leq \xi \leq \epsilon} |g''(A-\xi)| < \infty$,

$$|g(A-u+z) - g(A)| \leq (u-z)|g'(A)| + \frac{1}{2}(u-z)^2 \varrho_{A,\epsilon} \leq u|g'(A)| + \frac{1}{2}u^2 \varrho_{A,\epsilon}. \tag{A.1}$$

Therefore, by (2.2),

$$\int_0^\epsilon \Pi(\mathrm{d}u) \int_0^u e^{-\Phi(q)z} |g(z+A-u) - g(A)| \mathrm{d}z \leq \int_0^\epsilon \Pi(\mathrm{d}u) \Big(u^2 |g'(A)| + \frac{1}{2} u^3 \varrho_{A,\epsilon} \Big) < \infty.$$

On the other hand, by (2.14),

$$\int_\epsilon^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi(q)z} |g(z+A-u) - g(A)| \mathrm{d}z \leq \frac{1}{\Phi(q)} \int_\epsilon^\infty \Pi(\mathrm{d}u) \max_{A-u \leq \zeta \leq A} |g(\zeta) - g(A)| < \infty.$$

Combining the above, the proof is complete.

A.2. **Proof of Lemma 3.1.** Proof of (3.5): Define $\varrho(A) := \int_0^\infty \Pi(\mathrm{d}u) q(A;u)$ with

$$q(A;u) := \int_A^{u+A} e^{-\Phi(q)y} [g(y-u) - g(A)] \mathrm{d}y, \quad u \geq 0.$$

By assumption, we can choose $0 < \epsilon < 1$ such that g is C^2 on $[A-\epsilon, A+\epsilon]$.

We choose $0 < \delta < \epsilon$ that satisfies (3.1) and fix $0 < c < \delta$. By the mean value theorem, there exists $\xi \in (0,c)$ such that

$$q'(A+\xi;u) = \frac{q(A+c;u) - q(A;u)}{c}.$$

Because, for every $z \in (A, A+c)$, we have

$$q'(z;u) = e^{-\Phi(q)z} \Big(g(z) - g(z-u) - \frac{1-e^{-\Phi(q)u}}{\Phi(q)} g'(z) \Big), \tag{A.2}$$

the Taylor expansion implies that, for every $0 < u < \delta$,

$$|q'(A+\xi;u)| \leq e^{-\Phi(q)(A+\xi)} \frac{u^2}{2} (\Phi(q)+1) \max_{0 \leq \zeta \leq u} |g''(A+\xi-\zeta)| \leq e^{-\Phi(q)A} \frac{u^2}{2} (\Phi(q)+1) \max_{A-\delta \leq \zeta \leq A+\delta} |g''(\zeta)|,$$

or uniformly in $c \in (0,\delta)$

$$\frac{|q(A+c;u) - q(A;u)|}{c} \leq e^{-\Phi(q)A} \frac{u^2}{2} (\Phi(q)+1) \max_{A-\delta \leq \zeta \leq A+\delta} |g''(\zeta)|.$$

Hence uniformly in $c \in (0,\delta)$ by (2.2)

$$\int_0^\delta \Pi(\mathrm{d}u)\frac{|q(A+c;u)-q(A;u)|}{c} \leq e^{-\Phi(q)A}\max_{A-\delta\leq\zeta\leq A+\delta}|g''(\zeta)|\frac{\Phi(q)+1}{2}\int_0^\delta u^2\Pi(\mathrm{d}u)<\infty.$$

On the other hand, by (A.2),

$$\begin{aligned}\int_\delta^\infty \Pi(\mathrm{d}u)\frac{|q(A+c;u)-q(A;u)|}{c}&\\ \leq e^{-\Phi(q)A}\left(\max_{0\leq\xi\leq\delta}\frac{|g'(A+\xi)|}{\Phi(q)}\Pi(\delta,\infty)+\int_\delta^\infty \Pi(\mathrm{d}u)\max_{0\leq\xi\leq\delta}|(g(A+\xi)-g(A+\xi-u)|\right),\end{aligned}$$

which is finite by (3.1) and how δ is chosen.

This allows us to apply the dominated convergence theorem, and we obtain

$$\begin{aligned}\lim_{c\downarrow 0}\frac{\varrho(A+c)-\varrho(A)}{c}&=\int_0^\infty \Pi(\mathrm{d}u)\lim_{c\downarrow 0}\frac{q(A+c;u)-q(A;u)}{c}\\ &=\int_0^\infty \Pi(\mathrm{d}u)q'(A;u)=e^{-\Phi(q)A}\int_0^\infty \Pi(\mathrm{d}u)\Big(g(A)-g(A-u)-g'(A)\frac{1-e^{-\Phi(q)A}}{\Phi(q)}\Big).\end{aligned}$$

The proof for the left-derivative is similar, and this completes the proof of (3.5).

Proof of (3.6): Define

$$\widetilde{q}(z;u,x):=\int_z^{(u+z)\wedge x}W^{(q)}(x-y)[g(y-u)-g(z)]\mathrm{d}y,\quad z\in\mathbb{R}\text{ and }u>0. \tag{A.3}$$

Then, by (2.17), we have $\varphi_{g,A}^{(q)}(x)=\int_0^\infty \Pi(\mathrm{d}u)\widetilde{q}(A;u,x)$. We use the same $0<\delta<\epsilon$ as in the proof of (3.5) above and fix c and ε such that

$$0<c<\delta\wedge\frac{x-A}{4}\quad\text{and}\quad 0<\varepsilon:=\Big[\frac{1}{2}(x-A)-\Big(\delta\wedge\frac{x-A}{4}\Big)\Big]\wedge\delta.$$

It is then clear that $0<c<\varepsilon\leq\delta$. We shall split

$$\begin{aligned}\int_0^\infty \Pi(\mathrm{d}u)\frac{|\widetilde{q}(A+c;u,x)-\widetilde{q}(A;u,x)|}{c}&\\ =\int_0^\varepsilon \Pi(\mathrm{d}u)\frac{|\widetilde{q}(A+c;u,x)-\widetilde{q}(A;u,x)|}{c}+\int_\varepsilon^\infty \Pi(\mathrm{d}u)\frac{|\widetilde{q}(A+c;u,x)-\widetilde{q}(A;u,x)|}{c},\end{aligned} \tag{A.4}$$

and show that these two terms on the right-hand side are bounded in c on $(0,\delta\wedge\frac{x-A}{4})$.

For every fixed $0<u<\varepsilon$, our assumptions imply that $\widetilde{q}(\cdot;u,x)$ is C^2 on $(A,A+c)$. By the mean value theorem, there exists $\xi\in(0,c)$ such that

$$\widetilde{q}'(A+\xi;u,x)=\frac{\widetilde{q}(A+c;u,x)-\widetilde{q}(A;u,x)}{c}.$$

Given z at which g is differentiable and also satisfying $u+z<x$, differentiating (A.3) obtains

$$\widetilde{q}'(z;u,x)=W^{(q)}(x-z)(g(z)-g(z-u))-g'(z)\int_z^{u+z}W^{(q)}(x-y)\mathrm{d}y.$$

Because $x-u-A-\xi > \frac{x-A}{4} > 0$ (or $u+(A+\xi) < x$) and g is differentiable at $A+\xi$,

$$\begin{aligned}
&\frac{|\widetilde{q}(A+c;u,x)-\widetilde{q}(A;u,x)|}{c} = |\widetilde{q}'(A+\xi;u,x)| \\
&= \left| W^{(q)}(x-A-\xi)[g(A+\xi)-g(A+\xi-u)] - g'(A+\xi)\int_{A+\xi}^{u+A+\xi} W^{(q)}(x-y)\mathrm{d}y \right| \\
&\le W^{(q)}(x-A-\xi)\left|g(A+\xi)-g(A+\xi-u)-ug'(A+\xi)\right| \\
&\quad + \left| g'(A+\xi)\int_{A+\xi}^{u+A+\xi}(W^{(q)}(x-A-\xi)-W^{(q)}(x-y))\mathrm{d}y \right| \\
&\le W^{(q)}(x-A)\left|g(A+\xi)-g(A+\xi-u)-ug'(A+\xi)\right| \\
&\quad + u|g'(A+\xi)|\left|W^{(q)}(x-A-\xi)-W^{(q)}(x-u-A-\xi)\right| \\
&\le f_1(A;u,x)+f_2(A;u,x)
\end{aligned}$$

where

$$\begin{aligned}
f_1(A;u,x) &:= W^{(q)}(x-A)\max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}|g(A+\zeta)-g(A+\zeta-u)-ug'(A+\zeta)|, \\
f_2(A;u,x) &:= u\max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}|g'(A+\zeta)|\max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}\left|W^{(q)}(x-A-\zeta)-W^{(q)}(x-u-A-\zeta)\right|.
\end{aligned}$$

First, $\int_0^\varepsilon \Pi(\mathrm{d}u)f_1(A;u,x)$ is finite because, for every $u\le\varepsilon$, we have $u\le\delta$ and

$$\max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}|g(A+\zeta)-g(A+\zeta-u)-ug'(A+\zeta)| \le \frac{u^2}{2}\max_{A-\delta\le\zeta\le A+\delta}|g''(\zeta)|,$$

which is Π-integrable over $(0,\varepsilon)$ by (2.2). On the other hand, by (2.11) and because $0\le\zeta\le\delta\wedge\frac{x-A}{4}$ implies $x-u-A-\zeta > \frac{x-A}{4} > 0$, we have

$$\begin{aligned}
&\left|W^{(q)}(x-A-\zeta)-W^{(q)}(x-u-A-\zeta)\right| \\
&= \left|e^{\Phi(q)(x-A-\zeta)}W_{\Phi(q)}(x-A-\zeta)-e^{\Phi(q)(x-u-A-\zeta)}W_{\Phi(q)}(x-u-A-\zeta)\right| \\
&\le \left|\frac{e^{\Phi(q)(x-A-\zeta)}-e^{\Phi(q)(x-u-A-\zeta)}}{\psi'(\Phi(q))}\right| + e^{\Phi(q)(x-u-A-\zeta)}\left|W_{\Phi(q)}(x-A-\zeta)-W_{\Phi(q)}(x-u-A-\zeta)\right| \\
&\le e^{\Phi(q)(x-A)}\Big(\frac{1-e^{-\Phi(q)u}}{\psi'(\Phi(q))}+u\max_{\frac{x-A}{4}\le y\le x-A}W'_{\Phi(q)}(y)\Big),
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^\varepsilon \Pi(\mathrm{d}u)f_2(A;u,x) \\
&\le \max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}|g'(A+\zeta)|\int_0^\varepsilon u\max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}\left|W^{(q)}(x-A-\zeta)-W^{(q)}(x-u-A-\zeta)\right|\Pi(\mathrm{d}u) \\
&\le \max_{0\le\zeta\le\delta\wedge\frac{x-A}{4}}|g'(A+\zeta)|e^{\Phi(q)(x-A)}\int_0^\varepsilon u\Big(\frac{1-e^{-\Phi(q)u}}{\psi'(\Phi(q))}+u\max_{\frac{x-A}{4}\le y\le x-A}W'_{\Phi(q)}(y)\Big)\Pi(\mathrm{d}u),
\end{aligned}$$

which is finite by (2.2).

We now obtain the bound for the second term of the right-hand side of (A.4). For every $u > \varepsilon$ (which implies $u > c$), we have

$$\frac{|\widetilde{q}(A+c;u,x) - \widetilde{q}(A;u,x)|}{c} \leq B_1(A,c;u,x) + B_2(A,c;u,x)$$

where

$$B_1(A,c;u,x) := \frac{1}{c}\left[\int_{(u+A)\wedge x}^{(u+A+c)\wedge x} W^{(q)}(x-y)|g(y-u) - g(A+c)|\mathrm{d}y + \int_A^{A+c} W^{(q)}(x-y)|g(y-u) - g(A)|\mathrm{d}y\right],$$

$$B_2(A,c;u,x) := \frac{|g(A+c) - g(A)|}{c}\int_{A+c}^{(u+A)\wedge x} W^{(q)}(x-y)\mathrm{d}y.$$

For the former, we have

$$\begin{aligned}B_1(A,c;u,x) &\leq 3W^{(q)}(x-A)\max_{A-u\leq z\leq A+c}|g(z) - g(A)| \\ &\leq 3W^{(q)}(x-A)\Big(\max_{A-u\leq z\leq A}|g(z) - g(A)| + \max_{0\leq\zeta\leq\delta}|g(A+\zeta) - g(A+\zeta-u)|\Big).\end{aligned}$$

Here the first inequality holds because $|g(y-u) - g(A+c)| \leq |g(y-u) - g(A)| + |g(A) - g(A+c)|$. For the second inequality, it holds trivially when the maximum is attained for some $A - u \leq z \leq A$. If it is attained at $z = A + l$ for some $0 < l \leq c$. Then, because $A - u \leq A + l - u \leq A$ (thanks to $c < u$) and $c < \delta$

$$\begin{aligned}\max_{A-u\leq z\leq A+c}|g(z) - g(A)| &\leq |g(A+l-u) - g(A)| + |g(A+l) - g(A+l-u)| \\ &\leq \max_{A-u\leq z\leq A}|g(z) - g(A)| + \max_{0\leq\zeta\leq\delta}|g(A+\zeta) - g(A+\zeta-u)|.\end{aligned}$$

For the latter, by the C^2 property of g in the neighborhood of A, how δ is chosen and $c < \delta$, we obtain

$$\begin{aligned}B_2(A,c;u,x) &\leq \frac{|g(A+c) - g(A)|}{c}\int_{A+c}^{x} W^{(q)}(x-y)\mathrm{d}y \\ &\leq \Big(|g'(A)| + \frac{\delta}{2}\max_{A\leq\zeta\leq A+\delta}|g''(\zeta)|\Big)\int_A^x e^{\Phi(q)(x-y)}W_{\Phi(q)}(x-y)\mathrm{d}y \\ &\leq \frac{1}{\Phi(q)\psi'(\Phi(q))}\Big(|g'(A)| + \frac{\delta}{2}\max_{A\leq\zeta\leq A+\delta}|g''(\zeta)|\Big)e^{\Phi(q)(x-A)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\varepsilon^\infty \Pi(\mathrm{d}u)\frac{|\widetilde{q}(A+c;u,x) - \widetilde{q}(A;u,x)|}{c} \\ &\leq 3W^{(q)}(x-A)\int_\varepsilon^\infty \Pi(\mathrm{d}u)\Big(\max_{A-u\leq z\leq A}|g(z) - g(A)| + \max_{0\leq\zeta\leq\delta}|g(A+\zeta) - g(A+\zeta-u)|\Big) \\ &\quad + \frac{1}{\Phi(q)\psi'(\Phi(q))}\Big(|g'(A)| + \frac{\delta}{2}\max_{A\leq\zeta\leq A+\delta}|g''(\zeta)|\Big)e^{\Phi(q)(x-A)}\Pi(\varepsilon,\infty),\end{aligned}$$

which is bounded in c on $(0, \delta \wedge \frac{x-A}{4})$ by (2.14) and (3.1).

Hence, by the dominated convergence theorem,

$$\lim_{c\downarrow 0}\frac{\varphi^{(q)}_{g,A+c}(x)-\varphi^{(q)}_{g,A}(x)}{c}=\int_0^\infty \Pi(\mathrm{d}u)\lim_{c\downarrow 0}\frac{\widetilde{q}(A+c;u,x)-\widetilde{q}(A;u,x)}{c}=\int_0^\infty \Pi(\mathrm{d}u)\widetilde{q}'(A;u,x)$$
$$=\int_0^\infty \Pi(\mathrm{d}u)\Big[W^{(q)}(x-A)(g(A)-g(A-u))-g'(A)\int_A^{(u+A)\wedge x}W^{(q)}(x-z)\mathrm{d}z\Big].$$

The result for the left-derivative can be proved in the same way.

A.3. **Proof of Lemma 3.5.** It is known as in [11] that $\sigma>0$ guarantees that $W_{\Phi(q)}$ is twice continuously differentiable and hence $W'_{\Phi(q)}$ is continuous on $(0,\infty)$. Furthermore, (2.12) implies $W'_{\Phi(q)}(0+)=\frac{2}{\sigma^2}<\infty$ and (2.11) implies $\lim_{x\uparrow\infty}W'_{\Phi(q)}(x)=0$. Therefore, there exists $L<\infty$ such that

$$L:=\sup_{x>0}W'_{\Phi(q)}(x).$$

Now for every fixed $c>0$

$$\text{(A.5)}\quad \frac{1}{c}\int_0^\infty \Pi(\mathrm{d}u)\Big|\int_0^{u\wedge(x+c-A)}W^{(q)}(x+c-z-A)(g(z+A-u)-g(A))\mathrm{d}z$$
$$-\int_0^{u\wedge(x-A)}W^{(q)}(x-z-A)(g(z+A-u)-g(A))\mathrm{d}z\Big|\le f_1(x,A,c)+f_2(x,A,c),$$

where

$$f_1(x,A,c):=\int_0^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge(x-A)}q(x,c,z,A)|g(z+A-u)-g(A)|\mathrm{d}z,$$
$$f_2(x,A,c):=\int_{x-A}^\infty \Pi(\mathrm{d}u)\int_{u\wedge(x-A)}^{u\wedge(x+c-A)}\frac{W^{(q)}(x+c-z-A)}{c}|g(z+A-u)-g(A)|\mathrm{d}z,$$
$$q(x,c,z,A):=\frac{W^{(q)}(x+c-z-A)-W^{(q)}(x-z-A)}{c}.$$

Because

$$q(x,c,z,A)=\frac{e^{\Phi(q)(x+c-z-A)}W_{\Phi(q)}(x+c-z-A)-e^{\Phi(q)(x-z-A)}W_{\Phi(q)}(x-z-A)}{c}$$
$$=e^{\Phi(q)(x-z-A)}\frac{(e^{\Phi(q)c}-1)W_{\Phi(q)}(x+c-z-A)+\big(W_{\Phi(q)}(x+c-z-A)-W_{\Phi(q)}(x-z-A)\big)}{c}$$
$$\le e^{\Phi(q)(x-z-A)}\Big(\frac{e^{\Phi(q)c}-1}{c\psi'(\Phi(q))}+L\Big),$$

we have

$$f_1(x,A,c)\le e^{\Phi(q)(x-A)}\Big(\frac{e^{\Phi(q)c}-1}{c\psi'(\Phi(q))}+L\Big)\overline{\rho}^{(q)}_{g,A}.$$

On the other hand,

$$\begin{aligned} f_2(x,A,c) &\le W^{(q)}(x+c-A)\int_{x-A}^{\infty}\Pi(\mathrm{d}u)\max_{A\wedge(x-u)\le y\le A\wedge(x+c-u)}|g(y)-g(A)| \\ &\le \frac{e^{\Phi(q)(x+c-A)}}{\psi'(\Phi(q))}\int_{x-A}^{\infty}\Pi(\mathrm{d}u)\max_{A-u\le y\le A}|g(y)-g(A)|. \end{aligned}$$

By (2.14) , these bounds on f_1 and f_2 also bound (A.5), and hence by the dominated convergence theorem and because $W^{(q)}(0)=0$,

$$\begin{aligned} \lim_{c\downarrow 0}\frac{\varphi_{g,A}^{(q)}(x+c)-\varphi_{g,A}^{(q)}(x)}{c} &= \int_0^\infty \Pi(\mathrm{d}u)\frac{\partial}{\partial x}\int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))\mathrm{d}z \\ &= \int_0^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)(g(z+A-u)-g(A))\mathrm{d}z. \end{aligned}$$

The left-derivative can be obtained in the same way. This proves (3.9).

For the proof of (3.10), we first show the following lemma.

Lemma A.1. *There exists a finite constant C_A independent of x such that*

$$\int_0^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)|g(z+A-u)-g(A)|\mathrm{d}z \le e^{\Phi(q)(x-A)}C_A, \quad x>A.$$

Proof. Define

$$\begin{aligned} \phi_1(x,A) &:= \int_0^1 \Pi(\mathrm{d}u)\int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)|g(z+A-u)-g(A)|\mathrm{d}z, \\ \phi_2(x,A) &:= \int_1^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)|g(z+A-u)-g(A)|\mathrm{d}z. \end{aligned}$$

It is sufficient to show there exist finite constants $C_{1,A}$ and $C_{2,A}$ independent of x such that

(1) $\phi_1(x,A)\le e^{\Phi(q)(x-A)}C_{1,A}$,
(2) $\phi_2(x,A)\le e^{\Phi(q)(x-A)}C_{2,A}$.

Choosing $0<\epsilon<1$ and $\varrho_{A,\epsilon}$ as in the proof of Lemma 2.3, we obtain by (A.1)

$$\begin{aligned} \phi_1(x,A) \le& \int_0^\epsilon \Pi(\mathrm{d}u)\Big(u|g'(A)|+\frac{1}{2}u^2\varrho_{A,\epsilon}\Big)\int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)\mathrm{d}z \\ &+2\max_{A-1\le\zeta\le A}|g(\zeta)|\int_\epsilon^1\Pi(\mathrm{d}u)\int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)\mathrm{d}z \\ =& \int_0^\epsilon \Pi(\mathrm{d}u)\Big(u|g'(A)|+\frac{1}{2}u^2\varrho_{A,\epsilon}\Big)\Big(W^{(q)}(x-A)-W^{(q)}((x-A-u)\vee 0)\Big) \\ &+2\max_{A-1\le\zeta\le A}|g(\zeta)|\int_\epsilon^1\Pi(\mathrm{d}u)\Big(W^{(q)}(x-A)-W^{(q)}((x-A-u)\vee 0)\Big). \end{aligned}$$

Because, by (2.11),

$$
\begin{aligned}
&W^{(q)}(x-A) - W^{(q)}((x-A-u)\vee 0)\\
&= e^{\Phi(q)(x-A)}\Big[\big(W_{\Phi(q)}(x-A) - W_{\Phi(q)}((x-A-u)\vee 0)\big) + \Big(1-e^{-\Phi(\zeta)(u\wedge(x-A))}\Big) W_{\Phi(q)}((x-A-u)\vee 0)\Big]\\
&\leq e^{\Phi(q)(x-A)}\Big(Lu + \frac{1-e^{-\Phi(q)u}}{\psi'(\Phi(q))}\Big),
\end{aligned}
$$

we have $\phi_1(x,A) \leq e^{\Phi(q)(x-A)} C_{1,A}$ with

$$
C_{1,A} := \int_0^{\epsilon} \Pi(\mathrm{d}u)\Big(u|g'(A)| + \frac{1}{2}u^2 \varrho_{A,\epsilon}\Big)\Big(Lu + \frac{1-e^{-\Phi(q)u}}{\psi'(\Phi(q))}\Big) + 2 \max_{A-1\leq\zeta\leq A} |g(\zeta)| \int_{\epsilon}^{1} \Pi(\mathrm{d}u)\Big(Lu + \frac{1-e^{-\Phi(q)u}}{\psi'(\Phi(q))}\Big),
$$

which is finite thanks to (2.2) and by applying the Taylor expansion to $\big(1-e^{-\Phi(q)u}\big)$ to the first integral. Hence (1) is obtained. The proof of the existence of $C_{2,A}$ that satisfies (2) is immediate because

$$
\begin{aligned}
\phi_2(x,A) &\leq \int_1^{\infty} \Pi(\mathrm{d}u) \max_{A-u\leq y\leq A} |g(y)-g(A)| \int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)\mathrm{d}z\\
&\leq W^{(q)}(x-A)\int_1^{\infty} \Pi(\mathrm{d}u) \max_{A-u\leq y\leq A} |g(y)-g(A)|,
\end{aligned}
$$

and $W^{(q)}(x-A) \leq e^{\Phi(q)(x-A)}/\psi'(\Phi(q))$ by (2.11). □

Now using the lemma above, we can interchange the limit via the dominated convergence theorem as $x \downarrow A$ in (3.9) and obtain (3.10). This completes the proof.

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